

ORBIFOLD JACOBIAN ALGEBRAS FOR EXCEPTIONAL UNIMODAL SINGULARITIES

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ABSTRACT. This note shows that the orbifold Jacobian algebra associated to each invertible polynomial defining an exceptional unimodal singularity is isomorphic to the (usual) Jacobian algebra of the Berglund–Hübsch transform of an invertible polynomial defining the strange dual singularity in the sense of Arnold.

1. INTRODUCTION

Exceptional unimodal singularities consist of 14 isolated hypersurface singularities — Q_{10} , Q_{11} , Q_{12} , S_{11} , S_{12} , U_{12} , Z_{11} , Z_{12} , Z_{13} , W_{12} , W_{13} , E_{12} , E_{13} and E_{14} in the Arnold’s notation (see [AGV85]). Arnold observed a ”strange duality” in this class of singularities, the Dolgachev numbers (a triple of algebraically defined positive integers) of one singularity are equal to the Garbriellov numbers (a triple of positive integers associated to a Coxeter–Dynkin diagram) of another one and vice versa. It is now naturally understood as one of mirror symmetry phenomena (cf. [ET11] and references therein).

Let the polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ be *invertible* (see Definition 2). For such polynomials f one can associate a priori new polynomial $\tilde{f} \in \mathbb{C}[x_1, \dots, x_N]$, that is also invertible, called *Berglund–Hübsch transpose* of f (see Section 2 for details).

For any two exceptional unimodal singularities that are strange dual by Arnold there is a particular choice of the polynomials f_1, f_2 representing them such that both polynomials are invertible and $f_1 = \tilde{f}_2$. This was first observed in [KY95], where the authors show the coincidence of the elliptic genera of dual pairs up to sign, and also plays an essential role in [ET11] for a precise formulation and generalization of Arnold’s strange duality. However, the choice of an invertible polynomial, representing an exceptional unimodal singularity is not unique in general (we list all possible choices of an invertible polynomial, representing an exceptional unimodal singularity in Table 1).

For an invertible polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ and its symmetry group G_f^{SL} (see Section 2), let $\text{Jac}(f, G_f^{\text{SL}})$ stand for the orbifold Jacobian algebra of the pair (f, G_f^{SL}) and $\text{Jac}(f)$ be the “usual” Jacobian (or local) algebra. We prove the following theorem.

Type	f (v1)	f (v2)	f (v3)	Strange dual
Q_{10}	$x^4 + y^3 + xz^2$	—	—	E_{14}
Q_{11}	$x^3y + y^3 + xz^2$	—	—	Z_{13}
Q_{12}	$x^3z + y^3 + xz^2$	$x^5 + y^3 + xz^2$	—	Q_{12}
S_{11}	$x^4 + y^2z + xz^2$	—	—	W_{13}
S_{12}	$x^3y + y^2z + xz^2$	—	—	S_{12}
U_{12}	$x^4 + y^3 + z^3$	$x^4 + y^3 + z^2y$	$x^4 + y^2z + z^2y$	U_{12}
Z_{11}	$x^5 + xy^3 + z^2$	—	—	E_{13}
Z_{12}	$yx^4 + xy^3 + z^2$	—	—	Z_{12}
Z_{13}	$x^3z + xy^3 + z^2$	$x^6 + y^3x + z^2$	—	Q_{11}
W_{12}	$x^5 + y^2z + z^2$	$x^5 + y^4 + z^2$	—	W_{12}
W_{13}	$yx^4 + y^2z + z^2$	$x^4y + y^4 + z^2$	—	S_{11}
E_{12}	$x^7 + y^3 + z^2$	—	—	E_{12}
E_{13}	$y^3 + yx^5 + z^2$	—	—	Z_{11}
E_{14}	$x^4z + y^3 + z^2$	$x^8 + y^3 + z^2$	—	Q_{10}

TABLE 1. This table shows all possible invertible polynomials, representing an exceptional unimodal singularity of a given type.

Theorem 1. *Let $f_1, f_2 \in \mathbb{C}[x, y, z]$ be invertible polynomials defining exceptional unimodal singularities (full list is given in Table 1). There exists a Frobenius algebra isomorphism*

$$\text{Jac}(f_1) = \text{Jac}(f_1, \{\text{id}\}) \cong \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$$

if and only if the associated singularities of f_1 and f_2 are strange dual to each other in the sense of Arnold. Here \tilde{f}_2 is the Berglund–Hübsch transpose of f_2 .

For a fixed singularity and different choices of the invertible polynomial f_2 representing it, the function \tilde{f}_2 can have different symmetry groups $G_{\tilde{f}_2}^{\text{SL}}$ and even different Milnor numbers. In particular for U_{12} one will get the symmetry groups $\{\text{id}\}$, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$ and Milnor numbers 12, 12, 15 by \tilde{f}_2 . The algebra $\text{Jac}(f_1)$ in Theorem 1 will still be the same up to isomorphism. Hence Theorem 1 shows many non-trivial isomorphism.

It is worth to mention that Theorem 1 is compatible with the mirror symmetry. Let $\mathcal{A}^{\text{FJRW}}(f_2, \langle g_{f_2} \rangle)$ stand for the so-called FJRW ring, the analogue of quantum cohomology ring associated to the pair $(f_2, \langle g_{f_2} \rangle)$ and $g_{f_2} = (\mathbf{e}[w_x/d], \mathbf{e}[w_y/d], \mathbf{e}[w_z/d])$ be the so-called exponential grading operator where w_x, w_y, w_z, d are weights of variables x, y, z and the polynomial $f_2 = f_2(x, y, z)$ (see Section 2 for the notation). An isomorphism of Frobenius algebras $\text{Jac}(f_1, \{\text{id}\}) \cong \mathcal{A}^{\text{FJRW}}(f_2, \langle g_{f_2} \rangle)$ is obtained in [KP+]. As a corollary to Theorem 1, we get

$$\mathcal{A}^{\text{FJRW}}(f_2, \langle g_{f_2} \rangle) \cong \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}),$$

which is expected classical mirror symmetry isomorphism.

It's important to note that similar results are obtained in an apparently different context, in the study of matrix factorizations [CRR16] and [NR16]. We expect that the Hochschild cohomology group of the category of G -equivariant matrix factorizations will naturally yield the relationship between theirs and ours. We hope to elaborate on this subject in the near future.

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2. ORBIFOLD JACOBIAN ALGEBRA OF AN INVERTIBLE POLYNOMIAL

2.1. Invertible polynomials. For a non-negative integer n and $f = f(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ a polynomial, the *Jacobian algebra* $\text{Jac}(f)$ of f is a \mathbb{C} -algebra defined as

$$\text{Jac}(f) = \mathbb{C}[x_1, \dots, x_n] \Big/ \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right). \quad (1)$$

If $\text{Jac}(f)$ is a finite-dimensional \mathbb{C} -algebra, then set $\mu_f := \dim_{\mathbb{C}} \text{Jac}(f)$ and call it the *Milnor number* of f . In particular, if $n = 0$ then $\text{Jac}(f) = \mathbb{C}$ and $\mu_f = 1$.

The *Hessian* of f is defined as

$$\text{hess}(f) := \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,n}. \quad (2)$$

In particular, if $n = 0$ then $\text{hess}(f) = 1$.

A polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ is called a *weighted homogeneous* polynomial if there are positive integers w_1, \dots, w_N and d such that

$$f(\lambda^{w_1} x_1, \dots, \lambda^{w_N} x_N) = \lambda^d f(x_1, \dots, x_N) \quad (3)$$

for all $\lambda \in \mathbb{C}^*$. A weighted homogeneous polynomial f is called *non-degenerate* if it has at most an isolated critical point at the origin in \mathbb{C}^N , equivalently, if the Jacobian algebra $\text{Jac}(f)$ of f is finite-dimensional.

Definition 2. A non-degenerate weighted homogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ is called *invertible* if the following conditions are satisfied.

- The number of variables ($= N$) coincides with the number of monomials in the polynomial f , namely,

$$f(x_1, \dots, x_N) = \sum_{i=1}^N c_i \prod_{j=1}^N x_j^{E_{ij}} \quad (4)$$

for some coefficients $c_i \in \mathbb{C}^*$ and non-negative integers E_{ij} for $i, j = 1, \dots, N$.

- The matrix $E := (E_{ij})$ is invertible over \mathbb{Q} .
- The polynomial f and the *Berglund–Hübsch transpose* \tilde{f} of f defined by

$$\tilde{f}(x_1, \dots, x_N) := \sum_{i=1}^N c_i \prod_{j=1}^N x_j^{E_{ji}} \quad (5)$$

are non-degenerate.

Definition 3. The *group of maximal diagonal symmetries* of an invertible polynomial $f(x_1, \dots, x_N)$ is defined as

$$G_f := \{(\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \mid f(\lambda_1 x_1, \dots, \lambda_N x_N) = f(x_1, \dots, x_N)\}. \quad (6)$$

We shall always identify G_f with the subgroup of diagonal matrices of $\mathrm{GL}(N; \mathbb{C})$. Set

$$G_f^{\mathrm{SL}} := G_f \cap \mathrm{SL}(N; \mathbb{C}). \quad (7)$$

Each element $g \in G_f$ has a unique expression of the form

$$g = \mathrm{diag} \left(\mathbf{e} \left[\frac{a_1}{r} \right], \dots, \mathbf{e} \left[\frac{a_N}{r} \right] \right) \quad \text{with } 0 \leq a_i < r, \quad (8)$$

where $\mathbf{e}[\alpha] := \exp(2\pi\sqrt{-1}\alpha)$ and r is the order of g . We use the notation $(a_1/r, \dots, a_N/r)$ for the element g . The *age* of g is defined as the rational number

$$\mathrm{age}(g) := \frac{1}{r} \sum_{i=1}^N a_i. \quad (9)$$

Note that the $\mathrm{age}(g)$ is an integer if $g \in G_f^{\mathrm{SL}}$.

2.2. Orbifold Jacobian algebra. Let $f = f(x_1, \dots, x_N)$ be an invertible polynomial and G a subgroup of G_f^{SL} . A G -*twisted Jacobian algebra* of f $\mathrm{Jac}'(f, G)$, which exists and is uniquely defined up to an isomorphism by [BTW16, Theorem 21], is given as follows.

As a \mathbb{C} -vector space, $\mathrm{Jac}'(f, G)$ is given by

$$\mathrm{Jac}'(f, G) = \bigoplus_{g \in G} \mathrm{Jac}(f^g) \tilde{v}_g, \quad (10)$$

where $f^g := f|_{\mathrm{Fix}(g)}$, $\mathrm{Fix}(g) \subseteq \mathbb{C}^N$ is the fixed locus of g and \tilde{v}_g is a generator (a formal letter) attached to each $g \in G$. Note that f^g is also an invertible polynomial and there is a surjective map $\mathrm{Jac}(f) \rightarrow \mathrm{Jac}(f^g)$ ([ET13, Proposition 5] and [BTW16, Proposition 7]). It is also important that $\mathrm{Jac}'(f, G)$ is equipped with a $\mathbb{Z}/2\mathbb{Z}$ -grading according to the parity of $N - N_g$, the codimension of the fixed locus $\mathrm{Fix}(g)$, for each $g \in G$.

We are now ready to introduce the product structure on $\mathrm{Jac}'(f, G)$. For simplicity, we assume that G is a cyclic group whose order is a prime number.

For each pair (g, h) of elements in G and $\phi(\mathbf{x}), \psi(\mathbf{x}) \in \text{Jac}(f)$, it is defined as follows:

- Suppose that $\text{Fix}(g) \cup \text{Fix}(h) \cup \text{Fix}(gh) = \mathbb{C}^N$. Then

$$[\phi(\mathbf{x})]\tilde{v}_g \circ [\psi(\mathbf{x})]\tilde{v}_h := (-1)^{\frac{1}{2}(N-N_g)(N-N_g-1)} \cdot \mathbf{e} \left[-\frac{1}{2}\text{age}(g) \right] \cdot [\phi(\mathbf{x})\psi(\mathbf{x})H_{g,h}]\tilde{v}_{gh}, \quad (11)$$

where $H_{g,h} \in \mathbb{C}[x_1, \dots, x_N]$ is defined by the following equation in $\text{Jac}(f^{gh})$

$$\frac{1}{\mu_{fg \cap h}}[\text{hess}(f^{g \cap h})H_{g,h}] = \frac{1}{\mu_{fgh}}[\text{hess}(f^{gh})], \quad (12)$$

and here $f^{g \cap h}$ is an invertible polynomial given by the restriction $f|_{\text{Fix}(g) \cap \text{Fix}(h)}$ of f to the locus $\text{Fix}(g) \cap \text{Fix}(h)$.

- Suppose that $\text{Fix}(g) \cup \text{Fix}(h) \cup \text{Fix}(gh) \neq \mathbb{C}^N$. Then $[\phi(\mathbf{x})]\tilde{v}_g \circ [\psi(\mathbf{x})]\tilde{v}_h := 0$.

This completes the definition of $\text{Jac}'(f, G)$. It is easy to see that $\tilde{v}_{\text{id}} = [1]\tilde{v}_{\text{id}}$ is the identity of $\text{Jac}'(f, G)$.

Note that we have a natural action of G on $\text{Jac}(f^g)$ for any $g \in G$ and that the product structure is invariant under the G -action.

Definition 4. Let f and G be as above. The G -invariant $\mathbb{Z}/2\mathbb{Z}$ -graded subalgebra $\text{Jac}(f, G) := (\text{Jac}'(f, G))^G$ is called the *orbifold Jacobian algebra* of (f, G) .

An important property of this algebra is the following

Proposition 5 ([BTW16]). *The algebra $\text{Jac}(f, G)$ is a $\mathbb{Z}/2\mathbb{Z}$ -graded commutative Frobenius algebra. Namely, there is an even non-degenerate pairing $\eta_{f,G}$ such that*

$$\eta_{f,G}(X \circ Y, Z) = \eta_{f,G}(X, Y \circ Z), \quad X, Y, Z \in \text{Jac}'(f, G), \quad (13)$$

$$\eta_{f,G}(\tilde{v}_{\text{id}}, [\text{hess}(f)]\tilde{v}_{\text{id}}) = |G| \cdot \mu_f. \quad (14)$$

3. PROOF OF THEOREM 1

The proof of Theorem 1 is done by direct calculation. In what follows let the notation be as in Theorem 1.

Skipping the trivial cases when $f_1 = \tilde{f}_2$ and $G_{\tilde{f}_2}^{\text{SL}} = \{\text{id}\}$, to prove the theorem we only need to show that $\text{Jac}(f_1) \cong \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ for each row of Table 2 on page 6.

Further, note that if $G_{\tilde{f}_2}^{\text{SL}} = \{1\}$ and f_1, \tilde{f}_2 do not coincide but belong to the same right-equivalence class, the proof follows since the Jacobian algebra is an invariant of the right-equivalence class. Therefore, it is enough to show the statement for each row of Table 3 on page 6.

Type of f_1	f_1	\tilde{f}_2	$G_{\tilde{f}_2}^{\text{SL}}$	Type of f_2
E ₁₄	$x_1^8 + x_2^3 + x_3^2$	$x_1^4 x_2 + x_2^2 + x_3^3$	{id}	Q ₁₀
Q ₁₀	$x_1^4 + x_2^3 + x_1 x_3^2$	$x_1^8 + x_2^3 + x_3^2$	$\langle (1/2, 0, 1/2) \rangle$	E ₁₄
Q ₁₁	$x_1^3 x_2 + x_2^3 + x_1 x_3^2$	$x_1^6 x_2 + x_2^3 + x_3^2$	$\langle (1/2, 0, 1/2) \rangle$	Z ₁₃
Q ₁₂	$x_2^3 + x_1^3 x_3 + x_1 x_3^2$	$x_1^5 x_2 + x_2^2 + x_3^3$	$\langle (1/2, 1/2, 0) \rangle$	Q ₁₂
Q ₁₂	$x_1^5 + x_2^3 + x_1 x_3^2$	$x_1^3 + x_2^3 x_3 + x_2 x_3^2$	{id}	Q ₁₂
Q ₁₂	$x_1^5 + x_2^3 + x_1 x_3^2$	$x_1^5 x_2 + x_2^2 + x_3^3$	$\langle (1/2, 1/2, 0) \rangle$	Q ₁₂
S ₁₁	$x_1^4 + x_2^2 x_3 + x_1 x_3^2$	$x_1^4 + x_1 x_2^4 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	W ₁₃
U ₁₂	$x_1^4 + x_2^3 + x_3^3$	$x_1^4 + x_2^3 + x_3^3$	$\langle (0, 2/3, 1/3) \rangle$	U ₁₂
U ₁₂	$x_1^4 + x_2^3 + x_3^3$	$x_1^4 + x_2^3 x_3 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	U ₁₂
U ₁₂	$x_1^4 + x_2^3 + x_3^3$	$x_1^4 + x_2^2 x_3 + x_2 x_3^2$	{id}	U ₁₂
U ₁₂	$x_1^4 + x_2^3 + x_2 x_3^2$	$x_1^4 + x_2^3 + x_3^3$	$\langle (0, 2/3, 1/3) \rangle$	U ₁₂
U ₁₂	$x_1^4 + x_2^3 + x_2 x_3^2$	$x_1^4 + x_2^3 x_3 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	U ₁₂
U ₁₂	$x_1^4 + x_2^3 + x_2 x_3^2$	$x_1^4 + x_2^2 x_3 + x_2 x_3^2$	{id}	U ₁₂
U ₁₂	$x_1^4 + x_2^2 x_3 + x_2 x_3^2$	$x_1^4 + x_2^3 + x_3^3$	$\langle (0, 2/3, 1/3) \rangle$	U ₁₂
U ₁₂	$x_1^4 + x_2^2 x_3 + x_2 x_3^2$	$x_1^4 + x_2^3 x_3 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	U ₁₂
W ₁₂	$x_1^5 + x_2^2 x_3 + x_3^2$	$x_1^5 + x_2^4 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	W ₁₂
W ₁₂	$x_1^5 + x_2^4 + x_3^2$	$x_1^5 + x_2^2 + x_2 x_3^2$	{id}	W ₁₂
W ₁₂	$x_1^5 + x_2^4 + x_3^2$	$x_1^5 + x_2^4 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	W ₁₂
W ₁₃ ,	$x_1^4 x_2 + x_2^4 + x_3^2$	$x_1^4 x_2 + x_2^2 x_3 + x_3^2$	{id}	S ₁₁
Z ₁₃ ,	$x_1^6 + x_1 x_2^3 + x_3^2$	$x_1^3 x_2 + x_2^2 + x_1 x_3^3$	{id}	Q ₁₁

TABLE 2. To prove Theorem 1, we need to show $\text{Jac}(f_1) \cong \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ for every row of this table.

Type of f_1	f_1	\tilde{f}_2	Group $G_{\tilde{f}_2}^{\text{SL}}$	Type of f_2
Q ₁₀	$x_1^4 + x_2^3 + x_1 x_3^2$	$x_1^8 + x_2^3 + x_3^2$	$\langle (1/2, 0, 1/2) \rangle$	E ₁₄
Q ₁₁	$x_1^3 x_2 + x_2^3 + x_1 x_3^2$	$x_1^6 x_2 + x_2^3 + x_3^2$	$\langle (1/2, 0, 1/2) \rangle$	Z ₁₃
Q ₁₂	$x_1^5 + x_2^3 + x_1 x_3^2$	$x_1^5 x_2 + x_2^2 + x_3^3$	$\langle (1/2, 1/2, 0) \rangle$	Q ₁₂
S ₁₁	$x_1^4 + x_2^2 x_3 + x_1 x_3^2$	$x_1^4 + x_1 x_2^4 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	W ₁₃
U ₁₂	$x_1^4 + x_2^3 + x_3^3$	$x_1^4 + x_2^3 + x_3^3$	$\langle (0, 2/3, 1/3) \rangle$	U ₁₂
U ₁₂	$x_1^4 + x_2^3 + x_2 x_3^2$	$x_1^4 + x_2^3 x_3 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	U ₁₂
W ₁₂	$x_1^5 + x_2^4 + x_3^2$	$x_1^5 + x_2^4 + x_3^2$	$\langle (0, 1/2, 1/2) \rangle$	W ₁₂

TABLE 3. It is enough to show $\text{Jac}(f_1) \cong \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ for every row of this table to prove Theorem 1.

3.1. Computations. From now on, we shall use the notation of Table 2 on page 6. In order to check that two algebras are isomorphic, we represent $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ as a quotient algebra of a polynomial ring in three variables. Namely, we will compute relations in $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ and show the existence of a surjective algebra homomorphism from $\text{Jac}(f_1)$, which turns out to be an isomorphism due to the dimension reason.

3.1.1. Q_{10} and E_{14} . For $\tilde{f}_2 = x_1^8 + x_2^3 + x_3^2$, and $G_{\tilde{f}_2}^{\text{SL}} = \langle g \rangle = \langle (\frac{1}{2}, 0, \frac{1}{2}) \rangle$, $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ is a 10-dimensional \mathbb{C} -vector space, whose basis can be chosen as

$$\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}}, [x_1^4]\tilde{v}_{\text{id}}, [x_1^6]\tilde{v}_{\text{id}}, [x_2]\tilde{v}_{\text{id}}, [x_1^2x_2]\tilde{v}_{\text{id}}, [x_1^4x_2]\tilde{v}_{\text{id}}, [x_1^6x_2]\tilde{v}_{\text{id}}, \tilde{v}_g, [x_2]\tilde{v}_g. \quad (15)$$

The only non-trivial non-zero products in $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$, calculated by (11), are given by

$$\begin{aligned} [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^2]\tilde{v}_{\text{id}} &= [x_1^4]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^4]\tilde{v}_{\text{id}} = [x_1^6]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}} \circ [x_2]\tilde{v}_{\text{id}} = [x_1^2x_2]\tilde{v}_{\text{id}}, \\ [x_1^4]\tilde{v}_{\text{id}} \circ [x_2]\tilde{v}_{\text{id}} &= [x_1^4x_2]\tilde{v}_{\text{id}}, [x_1^6]\tilde{v}_{\text{id}} \circ [x_2]\tilde{v}_{\text{id}} = [x_1^6x_2]\tilde{v}_{\text{id}}, [x_2]\tilde{v}_{\text{id}} \circ \tilde{v}_g = [x_2]\tilde{v}_g, \\ [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^2x_2]\tilde{v}_{\text{id}} &= [x_1^4x_2]\tilde{v}_{\text{id}}, [x_1^4]\tilde{v}_{\text{id}} \circ [x_1^2x_2]\tilde{v}_{\text{id}} = [x_1^6x_2]\tilde{v}_{\text{id}}, \\ [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^4x_2]\tilde{v}_{\text{id}} &= [x_1^6x_2]\tilde{v}_{\text{id}}, \tilde{v}_g^2 = 16[x_1^6]\tilde{v}_{\text{id}}, \tilde{v}_g \circ [x_2]\tilde{v}_g = 16[x_1^6x_2]\tilde{v}_{\text{id}}, \end{aligned}$$

which show that $[x_1^2]\tilde{v}_{\text{id}}, [x_2]\tilde{v}_{\text{id}}, \tilde{v}_g$ generate $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ and are subject to the following relations

$$16([x_1^2]\tilde{v}_{\text{id}})^{\circ 3} - \tilde{v}_g^{\circ 2} = 0, \quad ([x_2]\tilde{v}_{\text{id}})^{\circ 2} = 0, \quad [x_1^2]\tilde{v}_{\text{id}} \circ \tilde{v}_g = 0. \quad (16)$$

On the other hand, the Jacobian algebra $\text{Jac}(f_1)$ is given by

$$\text{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3] / (4y_1^3 + y_3^2, y_2^2, y_1y_3). \quad (17)$$

Therefore, we have an algebra isomorphism

$$\text{Jac}(f_1) \xrightarrow{\cong} \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}), \quad y_1 \mapsto [x_1^2]\tilde{v}_{\text{id}}, \quad y_2 \mapsto [x_2]\tilde{v}_{\text{id}}, \quad y_3 \mapsto \frac{1}{2}\sqrt{-1}\tilde{v}_g, \quad (18)$$

which is, moreover, an isomorphism of Frobenius algebras since by (14) we have

$$\eta_{\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}}(\tilde{v}_{\text{id}}, [x_1^6x_2]\tilde{v}_{\text{id}}) = \frac{1}{672} \cdot \eta_{\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}}(\tilde{v}_{\text{id}}, [\text{hess}(\tilde{f}_2)]\tilde{v}_{\text{id}}) = \frac{2 \cdot 14}{672} = \frac{1}{24}, \quad (19)$$

$$\eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [y_1^3y_2]\tilde{v}_{\text{id}}) = \frac{1}{240} \cdot \eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [\text{hess}(f_1)]\tilde{v}_{\text{id}}) = \frac{1 \cdot 10}{240} = \frac{1}{24}. \quad (20)$$

3.1.2. Q_{11} and Z_{13} . For $\tilde{f}_2 = x_1^6x_2 + x_2^3 + x_3^2$, and $G_{\tilde{f}_2}^{\text{SL}} = \langle g \rangle = \langle (\frac{1}{2}, 0, \frac{1}{2}) \rangle$, $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ is a 11-dimensional \mathbb{C} -vector space, whose basis can be chosen as

$$\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}}, [x_1^4]\tilde{v}_{\text{id}}, [x_2]\tilde{v}_{\text{id}}, [x_1^2x_2]\tilde{v}_{\text{id}}, [x_1^4x_2]\tilde{v}_{\text{id}}, [x_2^2]\tilde{v}_{\text{id}}, [x_1^2x_2^2]\tilde{v}_{\text{id}}, [x_1^4x_2^2]\tilde{v}_{\text{id}}, \tilde{v}_g, [x_2]\tilde{v}_g. \quad (21)$$

The only non-trivial non-zero products in $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$, calculated by (11), are given by

$$\begin{aligned} [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^2]\tilde{v}_{\text{id}} &= [x_1^4]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^4]\tilde{v}_{\text{id}} = -3[x_2^2]\tilde{v}_{\text{id}}, [x_1^4]\tilde{v}_{\text{id}} \circ [x_1^4]\tilde{v}_{\text{id}} = -3[x_1^2x_2^2]\tilde{v}_{\text{id}}, \\ [x_1^2]\tilde{v}_{\text{id}} \circ [x_2]\tilde{v}_{\text{id}} &= [x_1^2x_2]\tilde{v}_{\text{id}}, [x_1^4]\tilde{v}_{\text{id}} \circ [x_2]\tilde{v}_{\text{id}} = [x_1^4x_2]\tilde{v}_{\text{id}}, [x_2]\tilde{v}_{\text{id}} \circ [x_2]\tilde{v}_{\text{id}} = [x_2^2]\tilde{v}_{\text{id}}, \\ [x_2]\tilde{v}_{\text{id}} \circ \tilde{v}_g &= [x_2]\tilde{v}_g, [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^2x_2]\tilde{v}_{\text{id}} = [x_1^4x_2]\tilde{v}_{\text{id}}, [x_2]\tilde{v}_{\text{id}} \circ [x_1^2x_2]\tilde{v}_{\text{id}} = [x_1^2x_2^2]\tilde{v}_{\text{id}}, \\ [x_1^2x_2]\tilde{v}_{\text{id}} \circ [x_1^2x_2]\tilde{v}_{\text{id}} &= [x_1^4x_2^2]\tilde{v}_{\text{id}}, [x_2]\tilde{v}_{\text{id}} \circ [x_1^4x_2]\tilde{v}_{\text{id}} = [x_1^4x_2^2]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = [x_1^2x_2^2]\tilde{v}_{\text{id}}, \\ [x_1^4]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} &= [x_1^4x_2^2]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^2x_2^2]\tilde{v}_{\text{id}} = [x_1^4x_2^2]\tilde{v}_{\text{id}}, \tilde{v}_g \circ \tilde{v}_g = 12[x_1^4x_2]\tilde{v}_{\text{id}}, \end{aligned}$$

$$\tilde{v}_g \circ [x_2]\tilde{v}_g = 12[x_1^4 x_2^2]\tilde{v}_{\text{id}}.$$

which show that $[x_1^2]\tilde{v}_{\text{id}}, [x_2]\tilde{v}_{\text{id}}, \tilde{v}_g$ generate $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ and are subject to the following relations

$$12([x_1^2]\tilde{v}_{\text{id}})^{\circ 2}[x_2]\tilde{v}_{\text{id}} - (\tilde{v}_g)^{\circ 2} = 0, ([x_1^2]\tilde{v}_{\text{id}})^{\circ 3} + 3([x_2]\tilde{v}_{\text{id}})^{\circ 2} = 0, [x_1^2]\tilde{v}_{\text{id}} \circ \tilde{v}_g = 0. \quad (22)$$

On the other hand, the Jacobian algebra $\text{Jac}(f_1)$ is given by

$$\text{Jac}(f) = \mathbb{C}[y_1, y_2, y_3] / (3y_1^2 y_2 + y_3^2, y_1^3 + 3y_2^2, 2y_1 y_3)$$

Therefore, we have an algebra isomorphism

$$\text{Jac}(f_1) \xrightarrow{\cong} \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}), \quad y_1 \mapsto [x_1^2]\tilde{v}_{\text{id}}, \quad y_2 \mapsto [x_2]\tilde{v}_{\text{id}}, \quad y_3 \mapsto \frac{1}{2}\sqrt{-1}\tilde{v}_g, \quad (23)$$

which is, moreover, an isomorphism of Frobenius algebras since by (14) we have

$$\eta_{\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}}(\tilde{v}_{\text{id}}, [x_1^4 x_2^2]\tilde{v}_{\text{id}}) = \frac{1}{18}, \quad (24)$$

$$\eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [y_1^2 y_2^2]\tilde{v}_{\text{id}}) = \frac{1}{198} \cdot \eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [\text{hess}(f_1)]\tilde{v}_{\text{id}}) = \frac{1 \cdot 11}{198} = \frac{1}{18}. \quad (25)$$

3.1.3. Q_{12} and Q_{12} . For $\tilde{f}_2 = x_1^5 x_2 + x_2^2 + x_3^3$, and $G_{\tilde{f}_2}^{\text{SL}} = \langle g \rangle = \langle (\frac{1}{2}, \frac{1}{2}, 0) \rangle$, $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ is a 12-dimensional \mathbb{C} -vector space, whose basis can be chosen as

$$\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}}, [x_1^4]\tilde{v}_{\text{id}}, [x_1 x_2]\tilde{v}_{\text{id}}, [x_1^3 x_2]\tilde{v}_{\text{id}}, [x_3]\tilde{v}_{\text{id}}, [x_1^2 x_3]\tilde{v}_{\text{id}}, \quad (26)$$

$$[x_1^4 x_3]\tilde{v}_{\text{id}}, [x_1 x_2 x_3]\tilde{v}_{\text{id}}, [x_1^3 x_2 x_3]\tilde{v}_{\text{id}}, \tilde{v}_g, [x_3]\tilde{v}_g. \quad (27)$$

The only non-trivial non-zero products in $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$, calculated by (11), are given by

$$\begin{aligned} [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^2]\tilde{v}_{\text{id}} &= [x_1^4]\tilde{v}_{\text{id}}, \quad [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^4]\tilde{v}_{\text{id}} = -2[x_1 x_2]\tilde{v}_{\text{id}}, \quad [x_1^4]\tilde{v}_{\text{id}} \circ x_1^4 \tilde{v}_{\text{id}} = -2[x_1^3 x_2]\tilde{v}_{\text{id}}, \\ [x_1^2]\tilde{v}_{\text{id}} \circ [x_1 x_2]\tilde{v}_{\text{id}} &= [x_1^3 x_2]\tilde{v}_{\text{id}}, \quad [x_1^2]\tilde{v}_{\text{id}} \circ [x_3]\tilde{v}_{\text{id}} = [x_1^2 x_3]\tilde{v}_{\text{id}}, \quad [x_1^4]\tilde{v}_{\text{id}} \circ [x_3]\tilde{v}_{\text{id}} = [x_1^4 x_3]\tilde{v}_{\text{id}}, \\ [x_1 x_2]\tilde{v}_{\text{id}} \circ [x_3]\tilde{v}_{\text{id}} &= [x_1 x_2 x_3]\tilde{v}_{\text{id}}, \quad [x_1^3 x_2]\tilde{v}_{\text{id}} \circ [x_3]\tilde{v}_{\text{id}} = [x_1^3 x_2 x_3]\tilde{v}_{\text{id}}, \quad [x_3]\tilde{v}_{\text{id}} \circ \tilde{v}_g = [x_3]\tilde{v}_g, \\ [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^2 x_3]\tilde{v}_{\text{id}} &= [x_1^4 x_3]\tilde{v}_{\text{id}}, \quad [x_1^4]\tilde{v}_{\text{id}} \circ [x_1^2 x_3]\tilde{v}_{\text{id}} = -2[x_1 x_2 x_3]\tilde{v}_{\text{id}}, \\ [x_1 x_2]\tilde{v}_{\text{id}} \circ [x_1^2 x_3]\tilde{v}_{\text{id}} &= [x_1^3 x_2 x_3]\tilde{v}_{\text{id}}, \quad [x_1^2]\tilde{v}_{\text{id}} \circ [x_1^4 x_3]\tilde{v}_{\text{id}} = -2[x_1 x_2 x_3]\tilde{v}_{\text{id}}, \\ [x_1^4]\tilde{v}_{\text{id}} \circ [x_1^4 x_3]\tilde{v}_{\text{id}} &= -2[x_1^3 x_2 x_3]\tilde{v}_{\text{id}}, \quad [x_1^2]\tilde{v}_{\text{id}} \circ [x_1 x_2 x_3]\tilde{v}_{\text{id}} = [x_1^3 x_2 x_3]\tilde{v}_{\text{id}}, \\ \tilde{v}_g \circ \tilde{v}_g &= 10[x_1^3 x_2]\tilde{v}_{\text{id}}, \quad \tilde{v}_g \circ [x_3]\tilde{v}_g = 10[x_1^3 x_2 x_3]\tilde{v}_{\text{id}}. \end{aligned}$$

which show that $[x_1^2]\tilde{v}_{\text{id}}, [x_3]\tilde{v}_{\text{id}}, \tilde{v}_g$ generate $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ and are subject to the following relations

$$[x_1^2]\tilde{v}_{\text{id}} \circ \tilde{v}_g = 0, \quad (\tilde{v}_g)^{\circ 2} - 5([x_1^2]\tilde{v}_{\text{id}})^{\circ 4} = 0, \quad ([x_3]\tilde{v}_{\text{id}})^{\circ 2} = 0 \quad (28)$$

On the other hand, the Jacobian algebra $\text{Jac}(f_1)$ is given by

$$\text{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3] / (5y_1^4 + y_3^2, 3y_2^2, 2y_1y_3). \quad (29)$$

Therefore, we have an algebra isomorphism

$$\text{Jac}(f_1) \xrightarrow{\cong} \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}), \quad y_1 \mapsto [x_1^2]\tilde{v}_{\text{id}}, \quad y_2 \mapsto \frac{1}{4}[x_3]\tilde{v}_{\text{id}}, \quad y_3 \mapsto \sqrt{-1}\tilde{v}_g. \quad (30)$$

which is, moreover, an isomorphism of Frobenius algebras since by (14) we have

$$\eta_{\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}}(\tilde{v}_{\text{id}}, [x_1^3x_2x_3]\tilde{v}_{\text{id}}) = \frac{1}{15}, \quad (31)$$

$$-\frac{4}{10} \cdot \eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [y_2y_3^2]\tilde{v}_{\text{id}}) = \frac{4}{720} \cdot \eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [\text{hess}(f_1)]\tilde{v}_{\text{id}}) = \frac{4 \cdot 12}{720} = \frac{1}{15}. \quad (32)$$

3.1.4. S_{11} and W_{13} . For $\tilde{f}_2 = x_1^4 + x_1x_2^4 + x_3^2$, and $G_{\tilde{f}_2}^{\text{SL}} = \langle g \rangle = \langle (0, \frac{1}{2}, \frac{1}{2}) \rangle$, $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ is a 11-dimensional \mathbb{C} -vector space, whose basis can be chosen as

$$\tilde{v}_{\text{id}}, [x_1]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}}, [x_1^3]\tilde{v}_{\text{id}}, [x_2^2]\tilde{v}_{\text{id}}, [x_1x_2^2]\tilde{v}_{\text{id}}, [x_1^3x_2^2]\tilde{v}_{\text{id}}, \quad \tilde{v}_g, [x_1]\tilde{v}_g, [x_1^2]\tilde{v}_g. \quad (33)$$

The only non-trivial non-zero products in $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$, calculated by (11), are given by

$$\begin{aligned} [x_1]\tilde{v}_{\text{id}} \circ [x_1]\tilde{v}_{\text{id}} &= [x_1^2]\tilde{v}_{\text{id}}, \quad [x_1]\tilde{v}_{\text{id}} \circ \tilde{v}_g = [x_1]\tilde{v}_g, \quad [x_1]\tilde{v}_{\text{id}} \circ [x_1]\tilde{v}_g = [x_1^2]\tilde{v}_g, \\ [x_1]\tilde{v}_{\text{id}} \circ [x_1^2]\tilde{v}_{\text{id}} &= [x_1^3]\tilde{v}_{\text{id}}, \quad [x_1^2]\tilde{v}_{\text{id}} \circ \tilde{v}_g = [x_1^2]\tilde{v}_g, \quad [x_1]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = [x_1x_2^2]\tilde{v}_{\text{id}}, \\ [x_1^2]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} &= [x_1^2x_2^2]\tilde{v}_{\text{id}}, \quad [x_1^3]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = [x_1^3x_2^2]\tilde{v}_{\text{id}}, \quad [x_2^2]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = -4[x_1^3]\tilde{v}_{\text{id}}, \\ [x_1]\tilde{v}_{\text{id}} \circ [x_1x_2^2]\tilde{v}_{\text{id}} &= [x_1^2x_2^2]\tilde{v}_{\text{id}}, \quad [x_1^2]\tilde{v}_{\text{id}} \circ [x_1x_2^2]\tilde{v}_{\text{id}} = [x_1^3x_2^2]\tilde{v}_{\text{id}}, \\ [x_1]\tilde{v}_{\text{id}} \circ [x_1^2x_2^2]\tilde{v}_{\text{id}} &= [x_1^3x_2^2]\tilde{v}_{\text{id}}, \quad \tilde{v}_g \circ \tilde{v}_g = 8[x_1x_2^2]\tilde{v}_{\text{id}}, \quad \tilde{v}_g \circ [x_1]\tilde{v}_g = 8[x_1^2x_2^2]\tilde{v}_{\text{id}}, \\ [x_1]\tilde{v}_g \circ [x_1]\tilde{v}_g &= 8[x_1^3x_2^2]\tilde{v}_{\text{id}}, \quad \tilde{v}_g \circ [x_1^2]\tilde{v}_g = 8[x_1^3x_2^2]\tilde{v}_{\text{id}}. \end{aligned}$$

which show that $[x_1]\tilde{v}_{\text{id}}, [x_2^2]\tilde{v}_{\text{id}}, \tilde{v}_g$ generate $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ and are subject to the following relations

$$([x_2^2]\tilde{v}_{\text{id}})^{\circ 2} + 4([x_1]\tilde{v}_{\text{id}})^{\circ 3} = 0, \quad [x_2^2]\tilde{v}_{\text{id}} \circ \tilde{v}_g = 0, \quad (\tilde{v}_g)^{\circ 2} - 8[x_1]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = 0. \quad (34)$$

On the other hand, the Jacobian algebra $\text{Jac}(f_1)$ is given by

$$\text{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3] / (4y_1^3 + y_3^2, 2y_2y_3, y_2^2 + 2y_1y_3). \quad (35)$$

Therefore, we have an algebra isomorphism

$$\text{Jac}(f_1) \xrightarrow{\cong} \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}), \quad y_1 \mapsto [x_1]\tilde{v}_{\text{id}}, \quad y_2 \mapsto \frac{1}{2}\sqrt{-1}\tilde{v}_g, \quad y_3 \mapsto [x_2^2]\tilde{v}_{\text{id}}. \quad (36)$$

which is, moreover, an isomorphism of Frobenius algebras since by (14) we have

$$\eta_{\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}}(\tilde{v}_{\text{id}}, [x_1^3 x_2^2] \tilde{v}_{\text{id}}) = \frac{1}{16}, \quad (37)$$

$$\eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [y_1^3 y_3] \tilde{v}_{\text{id}}) = \frac{1}{176} \cdot \eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [\text{hess}(f_1)] \tilde{v}_{\text{id}}) = \frac{1 \cdot 11}{176} = \frac{1}{16}. \quad (38)$$

3.1.5. U_{12} and U_{12} , part 1. For $\tilde{f}_2 = x_1^4 + x_2^3 + x_3^3$, and $G_{\tilde{f}_2}^{\text{SL}} = \langle g \rangle = \langle (0, \frac{2}{3}, \frac{1}{3}) \rangle$, $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ is a 12-dimensional \mathbb{C} -vector space, whose basis can be chosen as

$$\tilde{v}_{\text{id}}, [x_1] \tilde{v}_{\text{id}}, [x_1^2] \tilde{v}_{\text{id}}, [x_2 x_3] \tilde{v}_{\text{id}}, [x_1 x_2 x_3] \tilde{v}_{\text{id}}, [x_1^2 x_2 x_3] \tilde{v}_{\text{id}}, \quad (39)$$

$$\tilde{v}_{g^2}, [x_1] \tilde{v}_{g^2}, [x_1^2] \tilde{v}_{g^2}, \tilde{v}_g, [x_1] \tilde{v}_g, [x_1^2] \tilde{v}_g. \quad (40)$$

The only non-trivial non-zero products in $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$, calculated by (11), are given by

$$\begin{aligned} [x_1] \tilde{v}_{\text{id}} \circ [x_1] \tilde{v}_{\text{id}} &= [x_1^2] \tilde{v}_{\text{id}}, [x_1] \tilde{v}_{\text{id}} \circ \tilde{v}_{g^2} = [x_1] \tilde{v}_{g^2}, [x_1] \tilde{v}_{\text{id}} \circ [x_1] \tilde{v}_{g^2} = [x_1^2] \tilde{v}_{g^2}, \\ [x_1] \tilde{v}_{\text{id}} \circ \tilde{v}_g &= [x_1] \tilde{v}_g, [x_1] \tilde{v}_{\text{id}} \circ [x_1] \tilde{v}_g = [x_1^2] \tilde{v}_g, [x_1^2] \tilde{v}_{\text{id}} \circ \tilde{v}_{g^2} = [x_1^2] \tilde{v}_{g^2}, \\ [x_1^2 \tilde{v}_{\text{id}}] \circ \tilde{v}_g &= [x_1^2] \tilde{v}_g, [x_1] \tilde{v}_{\text{id}} \circ [x_2 x_3] \tilde{v}_{\text{id}} = [x_1 x_2 x_3] \tilde{v}_{\text{id}}, [x_1^2] \tilde{v}_{\text{id}} \circ [x_2 x_3] \tilde{v}_{\text{id}} = [x_1^2 x_2 x_3] \tilde{v}_{\text{id}}, \\ [x_1] \tilde{v}_{\text{id}} \circ [x_1 x_2 x_3] \tilde{v}_{\text{id}} &= [x_1^2 x_2 x_3] \tilde{v}_{\text{id}}, \tilde{v}_g \circ \tilde{v}_{g^2} = 9[x_2 x_3] \tilde{v}_{\text{id}}, \tilde{v}_g \circ [x_1] \tilde{v}_{g^2} = 9[x_1 x_2 x_3] \tilde{v}_{\text{id}}, \\ \tilde{v}_g \circ [x_1^2] \tilde{v}_{g^2} &= 9[x_1^2 x_2 x_3] \tilde{v}_{\text{id}}, [x_1] \tilde{v}_g \circ \tilde{v}_{g^2} = 9[x_1 x_2 x_3] \tilde{v}_{\text{id}}, [x_1] \tilde{v}_g \circ [x_1] \tilde{v}_{g^2} = 9[x_1^2 x_2 x_3] \tilde{v}_{\text{id}}, \\ [x_1^2] \tilde{v}_g \circ \tilde{v}_{g^2} &= 9[x_1^2 x_2 x_3] \tilde{v}_{\text{id}}. \end{aligned}$$

which show that $[x_1] \tilde{v}_{\text{id}}, \tilde{v}_g, \tilde{v}_{g^2}$ generate $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ and are subject to the following relations

$$([x_1] \tilde{v}_{\text{id}})^{\circ 3} = 0, (\tilde{v}_g)^{\circ 2} = 0, (\tilde{v}_{g^2})^{\circ 2} = 0. \quad (41)$$

On the other hand, the Jacobian algebra $\text{Jac}(f_1)$ is given by

$$\text{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3] / (4y_1^3, 3y_2^2, 3y_3^2). \quad (42)$$

Therefore, we have an algebra isomorphism

$$\text{Jac}(f_1) \xrightarrow{\cong} \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}), \quad y_1 \mapsto \frac{1}{3}[x_1] \tilde{v}_{\text{id}}, \quad y_2 \mapsto \frac{1}{\sqrt{3}} \tilde{v}_g, \quad y_3 \mapsto \frac{1}{\sqrt{3}} \tilde{v}_{g^2}. \quad (43)$$

which is, moreover, an isomorphism of Frobenius algebras since by (14) we have

$$\eta_{\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}}(\tilde{v}_{\text{id}}, [x_1^2 x_2 x_3] \tilde{v}_{\text{id}}) = \frac{1}{12}, \quad (44)$$

$$3 \cdot \eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [y_1^2 y_2 y_3] \tilde{v}_{\text{id}}) = \frac{3}{432} \cdot \eta_{f_1, \{\text{id}\}}(\tilde{v}_{\text{id}}, [\text{hess}(f_1)] \tilde{v}_{\text{id}}) = \frac{3 \cdot 12}{432} = \frac{1}{12}. \quad (45)$$

3.1.6. U_{12} and U_{12} , part 2. For $\tilde{f}_2 = x_1^4 + x_2^3 x_3 + x_3^2$, and $G_{\tilde{f}_2}^{\text{SL}} = \langle g \rangle = \langle (0, \frac{1}{2}, \frac{1}{2}) \rangle$, $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ is a 12-dimensional \mathbb{C} -vector space, whose basis can be chosen as

$$\tilde{v}_{\text{id}}, [x_1]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}}, [x_2^2]\tilde{v}_{\text{id}}, [x_1 x_2^2]\tilde{v}_{\text{id}}, [x_1^2 x_2^2]\tilde{v}_{\text{id}}, [x_2 x_3]\tilde{v}_{\text{id}}, [x_1 x_2 x_3]\tilde{v}_{\text{id}}, [x_1^2 x_2 x_3]\tilde{v}_{\text{id}}, \quad (46)$$

$$\tilde{v}_g, [x_1]\tilde{v}_g, [x_1^2]\tilde{v}_g. \quad (47)$$

The only non-trivial non-zero products in $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$, calculated by (11), are given by

$$\begin{aligned} [x_1]\tilde{v}_{\text{id}} \circ [x_1]\tilde{v}_{\text{id}} &= [x_1^2]\tilde{v}_{\text{id}}, \quad [x_1]\tilde{v}_{\text{id}} \circ \tilde{v}_g = [x_1]\tilde{v}_g, \quad [x_1]\tilde{v}_{\text{id}} \circ [x_1]\tilde{v}_g = [x_1^2]\tilde{v}_g, \\ [x_1^2]\tilde{v}_{\text{id}} \circ \tilde{v}_g &= [x_1^2]\tilde{v}_g, \quad [x_1]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = [x_1 x_2^2]\tilde{v}_{\text{id}}, \quad [x_1^2]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = [x_1^2 x_2^2]\tilde{v}_{\text{id}}, \\ [x_2^2]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} &= -2[x_2 x_3]\tilde{v}_{\text{id}}, \quad [x_1]\tilde{v}_{\text{id}} \circ [x_1 x_2^2]\tilde{v}_{\text{id}} = [x_1^2 x_2^2]\tilde{v}_{\text{id}}, \\ [x_2^2]\tilde{v}_{\text{id}} \circ [x_1 x_2^2]\tilde{v}_{\text{id}} &= -2[x_1 x_2 x_3]\tilde{v}_{\text{id}}, \quad [x_1 x_2^2]\tilde{v}_{\text{id}} \circ [x_1 x_2^2]\tilde{v}_{\text{id}} = -2[x_1^2 x_2 x_3]\tilde{v}_{\text{id}}, \\ [x_2^2]\tilde{v}_{\text{id}} \circ [x_1^2 x_2^2]\tilde{v}_{\text{id}} &= -2[x_1^2 x_2 x_3]\tilde{v}_{\text{id}}, \quad [x_1]\tilde{v}_{\text{id}} \circ [x_2 x_3]\tilde{v}_{\text{id}} = [x_1 x_2 x_3]\tilde{v}_{\text{id}}, \\ [x_1^2]\tilde{v}_{\text{id}} \circ [x_2 x_3]\tilde{v}_{\text{id}} &= [x_1^2 x_2 x_3]\tilde{v}_{\text{id}}, \quad [x_1]\tilde{v}_{\text{id}} \circ [x_1 x_2 x_3]\tilde{v}_{\text{id}} = [x_1^2 x_2 x_3]\tilde{v}_{\text{id}}, \\ \tilde{v}_g \circ \tilde{v}_g &= 6[x_2 x_3]\tilde{v}_{\text{id}}, \quad \tilde{v}_g \circ [x_1]\tilde{v}_g = 6[x_1 x_2 x_3]\tilde{v}_{\text{id}}, \\ [x_1]\tilde{v}_g \circ [x_1]\tilde{v}_g &= 6[x_1^2 x_2 x_3]\tilde{v}_{\text{id}}, \quad \tilde{v}_g \circ [x_1^2]\tilde{v}_g = 6[x_1^2 x_2 x_3]\tilde{v}_{\text{id}}. \end{aligned}$$

which show that $[x_1]\tilde{v}_{\text{id}}, [x_2^2]\tilde{v}_{\text{id}}, \tilde{v}_g$ generate $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ and are subject to the following relations

$$([x_1]\tilde{v}_{\text{id}})^{\circ 3} = 0, \quad (\tilde{v}_g)^{\circ 2} + 3([x_2^2]\tilde{v}_{\text{id}})^{\circ 2}, \quad [x_2^2]\tilde{v}_{\text{id}} \circ \tilde{v}_g = 0. \quad (48)$$

On the other hand, the Jacobian algebra $\text{Jac}(f_1)$ is given by

$$\text{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3] / (4y_1^3, 3y_2^2 + y_3^2, 2y_2 y_3). \quad (49)$$

Therefore, we have an algebra isomorphism

$$\text{Jac}(f_1) \xrightarrow{\cong} \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}), \quad y_1 \mapsto \frac{1}{\sqrt{-6}}[x_1]\tilde{v}_{\text{id}}, \quad y_2 \mapsto [x_2^2]\tilde{v}_{\text{id}}, \quad y_3 \mapsto \tilde{v}_g. \quad (50)$$

which is, moreover, an isomorphism of Frobenius algebras since by (14) we have

$$\eta_{\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}}(\tilde{v}_{\text{id}}, [x_1^2 x_2 x_3]\tilde{v}_{\text{id}}) = \frac{1}{12}, \quad (51)$$

$$\frac{6}{9} \cdot \eta_{f_1, \{ \text{id} \}}(\tilde{v}_{\text{id}}, [y_1^2 y_3^2]\tilde{v}_{\text{id}}) = \frac{6}{9 \cdot 96} \cdot \eta_{f_1, \{ \text{id} \}}(\tilde{v}_{\text{id}}, [\text{hess}(f_1)]\tilde{v}_{\text{id}}) = \frac{6 \cdot 12}{9 \cdot 96} = \frac{1}{12}. \quad (52)$$

3.1.7. W_{12} and W_{12} . For $\tilde{f}_2 = x_1^5 + x_2^4 + x_3^2$, and $G_{\tilde{f}_2}^{\text{SL}} = \langle g \rangle = \langle (0, \frac{1}{2}, \frac{1}{2}) \rangle$, $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ is a 12-dimensional \mathbb{C} -vector space, whose basis can be chosen as

$$\tilde{v}_{\text{id}}, [x_1]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}}, [x_1^3]\tilde{v}_{\text{id}}, [x_2^2]\tilde{v}_{\text{id}}, [x_1 x_2^2]\tilde{v}_{\text{id}}, [x_1^2 x_2^2]\tilde{v}_{\text{id}}, [x_1^3 x_2^2]\tilde{v}_{\text{id}}, \quad \tilde{v}_g, [x_1]\tilde{v}_g, [x_1^2]\tilde{v}_g, [x_1^3]\tilde{v}_g. \quad (53)$$

The only non-trivial non-zero products in $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$, calculated by (11), are given by

$$\begin{aligned}
[x_1]\tilde{v}_{\text{id}} \circ [x_1]\tilde{v}_{\text{id}} &= [x_1^2]\tilde{v}_{\text{id}}, [x_1]\tilde{v}_{\text{id}} \circ \tilde{v}_g = [x_1]\tilde{v}_g, [x_1]\tilde{v}_{\text{id}} \circ [x_1]\tilde{v}_g = [x_1^2]\tilde{v}_g, \\
[x_1]\tilde{v}_{\text{id}} \circ [x_1^2]\tilde{v}_g &= [x_1^3]\tilde{v}_g, [x_1]\tilde{v}_{\text{id}} \circ [x_1^2]\tilde{v}_{\text{id}} = [x_1^3]\tilde{v}_{\text{id}}, [x_1^2]\tilde{v}_{\text{id}} \circ \tilde{v}_g = [x_1^2]\tilde{v}_g, \\
[x_1^2]\tilde{v}_{\text{id}} \circ [x_1]\tilde{v}_g &= [x_1^3]\tilde{v}_g, [x_1^3]\tilde{v}_{\text{id}} \circ \tilde{v}_g = [x_1^3]\tilde{v}_g, [x_1]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = [x_1x_2^2]\tilde{v}_{\text{id}}, \\
[x_1^2]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} &= [x_1^2x_2^2]\tilde{v}_{\text{id}}, [x_1^3]\tilde{v}_{\text{id}} \circ [x_2^2]\tilde{v}_{\text{id}} = [x_1^3x_2^2]\tilde{v}_{\text{id}}, [x_1]\tilde{v}_{\text{id}} \circ [x_1x_2^2]\tilde{v}_{\text{id}} = [x_1^2x_2^2]\tilde{v}_{\text{id}}, \\
[x_1^2]\tilde{v}_{\text{id}} \circ [x_1x_2^2]\tilde{v}_{\text{id}} &= [x_1^3x_2^2]\tilde{v}_{\text{id}}, [x_1]\tilde{v}_{\text{id}} \circ [x_1^2x_2^2]\tilde{v}_{\text{id}} = [x_1^3x_2^2]\tilde{v}_{\text{id}}, \tilde{v}_g \circ \tilde{v}_g = 8[x_2^2]\tilde{v}_{\text{id}}, \\
\tilde{v}_g \circ [x_1]\tilde{v}_g &= 8[x_1x_2^2]\tilde{v}_{\text{id}}, [x_1]\tilde{v}_g \circ [x_1]\tilde{v}_g = 8[x_1^2x_2^2]\tilde{v}_{\text{id}}, \tilde{v}_g \circ [x_1^2]\tilde{v}_g = 8[x_1^2x_2^2]\tilde{v}_{\text{id}}, \\
[x_1]\tilde{v}_g \circ [x_1^2]\tilde{v}_g &= 8[x_1^3x_2^2]\tilde{v}_{\text{id}}, \tilde{v}_g \circ [x_1^3]\tilde{v}_g = 8[x_1^3x_2^2]\tilde{v}_{\text{id}}.
\end{aligned}$$

which show that $[x_1]\tilde{v}_{\text{id}}$ and \tilde{v}_g generate $\text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}})$ and are subject to the following relations

$$([x_1]\tilde{v}_{\text{id}})^{\circ 4} = 0, (\tilde{v}_g)^{\circ 3} = 0 \quad (54)$$

On the other hand, the Jacobian algebra $\text{Jac}(f_1)$ is given by

$$\text{Jac}(f_1) = \mathbb{C}[y_1, y_2, y_3] / (5y_1^4, 4y_2^3, y_3). \quad (55)$$

Therefore, we have an algebra isomorphism

$$\text{Jac}(f_1) \xrightarrow{\cong} \text{Jac}(\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}), \quad y_1 \mapsto \frac{1}{2}[x_1]\tilde{v}_{\text{id}}, \quad y_2 \mapsto \frac{1}{\sqrt{2}}\tilde{v}_g. \quad (56)$$

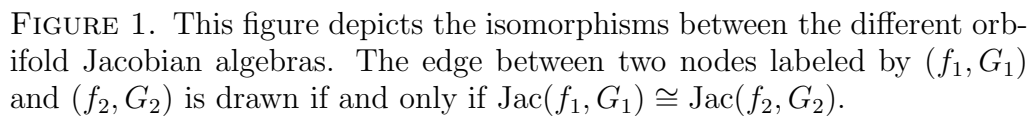
which is, moreover, an isomorphism of Frobenius algebras since by (14) we have

$$\begin{aligned}
\eta_{\tilde{f}_2, G_{\tilde{f}_2}^{\text{SL}}}(\tilde{v}_{\text{id}}, [x_1^3x_2^2]\tilde{v}_{\text{id}}) &= \frac{1}{20}, \\
\frac{16}{8} \cdot \eta_{f_1, \{id\}}(\tilde{v}_{\text{id}}, [y_1^3y_2^2]\tilde{v}_{\text{id}}) &= \frac{16}{8 \cdot 480} \cdot \eta_{f_1, \{id\}}(\tilde{v}_{\text{id}}, [\text{hess}(f_1)]\tilde{v}_{\text{id}}) = \frac{16 \cdot 12}{480} = \frac{1}{20}.
\end{aligned} \quad (57)$$

3.2. Remark. In order to visualize the statement of Theorem 1 consider the following Figure 1 on page 13. The nodes of this figure are the pairs (f, G) where f is an invertible polynomial and $G \subseteq G_f^{\text{SL}}$. The edge between two nodes labeled by (f_1, G_1) and (f_2, G_2) respectively is drawn if and only if $\text{Jac}(f_1, G_1) \cong \text{Jac}(f_2, G_2)$. All the pairs (f, G) considered are those from Table 2.

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